Absence of Two-Dimensional Bragg Glasses

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The stability to dislocations of the elastic phase, or "Bragg glass", of a randomly pinned elastic medium in two dimensions is studied using the minimum-cost-flow algorithm for a disordered fully-packed loop model. The elastic phase is found to be unstable to dislocations due to the quenched disorder. The energetics of dislocations are discussed within the framework of renormalization group predictions as well as in terms of a domain wall picture.

PACS number: 74.60.Ge, 64.70.Pf, 02.60.Pn

Randomly pinned elastic media are used to model various condensed-matter systems with quenched disorder, including flux-line arrays in dirty type-II superconductors [1] and charge density waves [2]. Although it is known [3] that these systems cannot exhibit long-range translational order in less than four dimensions, the intriguing possibility of a "topologically ordered" low-temperature phase remains an open question [4–6]. It has been conjectured that such a phase, with all dislocation loops bound, exists in three dimensions. Such a phase would be elastic and have power-law Bragg-like singularities in its structure factor; it is often referred to as a "Bragg" or "elastic" glass [4,6].

Whether or not unbound topological defects exist at low temperatures involves a subtle balance between elastic-energy cost and disorder-energy gain [5]. In this letter, we analyze this issue for two-dimensional randomly pinned elastic media at zero temperature by considering a $2d$-lattice model, viz., a fully-packed loop (FPL) model [7] with quenched disorder. Exact ground states of the FPL model are computed with and without topological defects, which we refer to as dislocations. The *polynomial* optimization algorithm [8] that we use, minimum-cost-flow, enables us to study large systems.

We focus on the energetics of a single dislocation pair in systems of size $L\times L$. Our main conclusion is that the disorder energy gain of the optimally placed pair dominates over the elastic energy cost with the results being consistent with theoretical predictions of $O(-\ln^{3/2} L)$ and $O(+\ln L)$ respectively for the two quantities. Dislocations therefore become unbound and proliferate causing the destruction of the Bragg glass.

Model and algorithm The FPL model is defined on a honeycomb lattice on which all configurations of occupied bonds which form closed loops and cover every site exactly once are allowed, as shown in Fig.1(a). This model can be mapped to a solid-on-solid surface model. Define integer heights at the centers of hexagons then orient all bonds of the resulting triangular lattice such that elementary triangles pointing upward are circled clockwise; assign 1 to the difference of neighboring heights along the oriented bonds if a loop is crossed and -2 otherwise. This yields single-valued heights up to an overall constant. It can be seen that the smallest "step" of the surface is three, so that the effective potential on the surface is periodic in heights modulo 3.

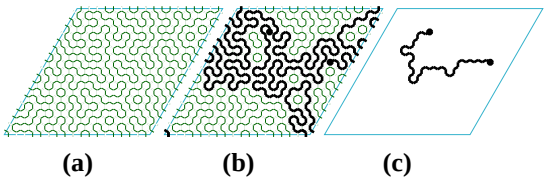

FIG. 1. The fully packed loop model with periodic boundary conditions imposed. The ground states with and without a pair of dislocations for one realization of random bond weights are displayed in (b) and (a) respectively. The dislocations (solid dots) in (b) are connected by an open string (thick line) among the loops. The relevant physical object is, however, the domain wall which is induced by the dislocations as shown in (c). This domain wall represents the line of bond *differences* between the ground states (a) and (b).

Quenched disorder is introduced via random bond weights on the honeycomb lattice, chosen independently and uniformly from integers in the interval [-500, 500]. The energy is the sum of the bond weights along all loops and strings. The system can be viewed as a surface in a $3d$ random medium that is periodic in the height direction. Dislocations are added to the FPL model by "violating" the constraint. One dislocation pair is an open string in an otherwise fully-packed system as shown in Fig.1(b). The height change along any path encircling one end of the string is the Burgers charge ± 3 of a dislocation so that the heights become multi-valued. Note that the configurations with and without a dislocation pair only differ on a domain "wall", as shown in Fig.1(c).

Finding ground states is equivalent to an integer minimum-cost flow problem on a suitably designed graph; details can be found elsewhere [8,9].

Numerical results For each disorder realization, the ground-state energies with and without a dislocation pair and hence the defect energy, E_d, and also the connecting wall (See Fig.1(c)) were computed.

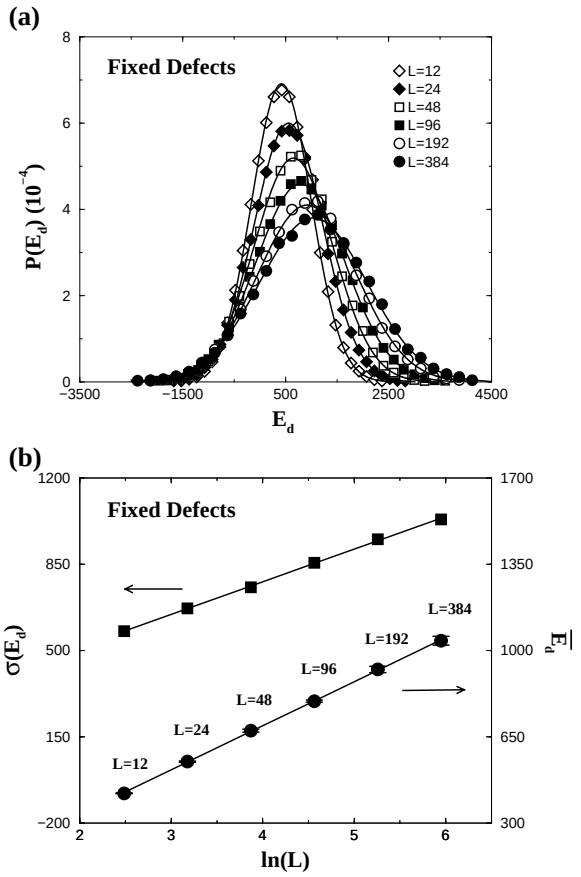

FIG. 2. Energetics of *fixed* defects: The probability distributions of the energy of a *fixed* dislocation pair with separation $L/2$ for sample sizes from $L = 12$ to $L = 384$ are shown in (a). The solid lines are gaussian fits. The corresponding average defect energy $\overline{E_d}$ (solid circle) and the root-mean-square (rms) width $\sigma(E_d)$ (solid square) are found to scale with system size as $\ln L$ as shown in (b). The solid lines are linear fits.

We first held the dislocations *fixed* at two specific sites separated by $L/2$ in $L \times L$ samples with $L = 12, 24, 48, 96, 192$, and 384, with at least 10^4 disorder realizations for each size. The probability distributions $P(E_d)$ of the defect energy are shown in Fig.2(a); they are well fit by gaussians for all sizes. The mean pair-energy is plotted versus $\ln L$ with a linear fit $\overline{E_d} = 180(2)\ln L - 20(6)$. The root-mean-square (rms) width $\sigma(E_d)$ is also shown in Fig.2(b), with a linear fit of $\sigma(E_d) = 250(3)+133(1)\ln L$. This implies a tail in $P(E_d)$ for negative E_d and suggests an energy *gain* from dislocations that can be optimized to take advantage of the negative part of the distribution.

We have computed the *optimized* (lowest energy) dislocation pair energy $E_d^{\min}$ for L up to 480 with $10^4 - 10^6$ samples for each size. The defect energy distribution $P(E_d^{\min})$ is no longer Gaussian, indeed substantial asymmetry in $P(E_d^{\min})$ is seen in Fig.3(a). Moreover, in contrast to the case of fixed dislocations, $\overline{E_d^{\min}}$ is *negative* and *decreases* more rapidly than $\ln L$ with increasing system size while the rms width $\sigma(E_d^{\min})$ increases less rapidly than $\ln L$. The linear fits shown in Fig.3(b) yield $[|\overline{E_d^{\min}}|]^{2/3} = (43.80 \pm 0.31)\ln(L) + (24.03 \pm 0.14)$ and $[\sigma(E_d^{\min})]^2 = (21883 \pm 180)\ln(L) + (9013 \pm 802)$. This implies that since almost all large systems have negative energy dislocation pairs, as is evident in Fig.3(a) the FPL model is *unstable* against the spontaneous appearance of dislocations.

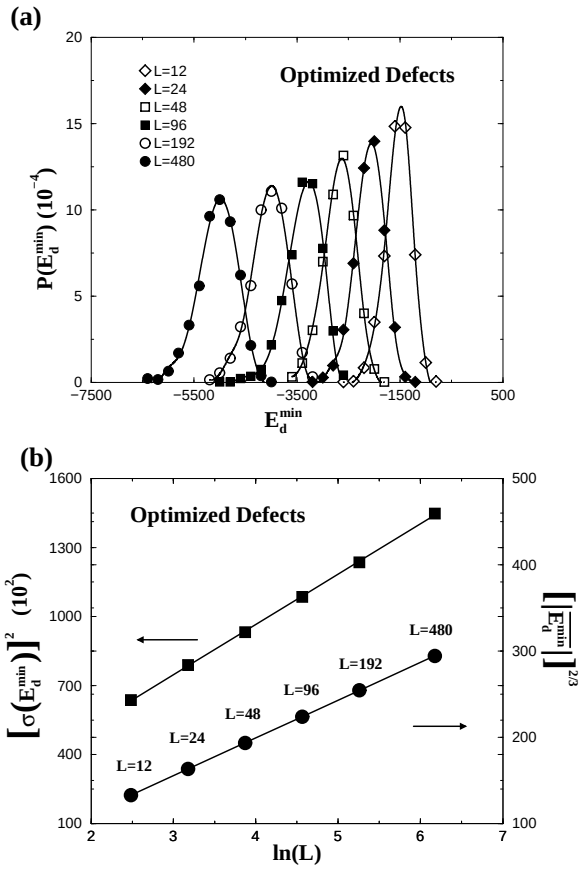

FIG. 3. Energetics of *optimized* defects: The defect energy probability distributions for sample sizes from $L = 12$ to $L = 480$ are shown in (a). The solid lines are guides to the eyes. Both the average defect energy plotted as $[|\overline{E_d^{\min}}|]^{2/3}$ vs. $\ln L$ and the rms width plotted as $[\sigma(E_d^{\min})]^2$ vs. $\ln L$ are shown in (b). Solid lines are linear fits.

Continuum models To shed some light on the observed defect energetics, and to make contact with analytic work, we consider coarse-grained continuum approximations to the random FPL model. In the absence of dislocations, the surface has a stiffness caused by the inability of a tilted and hence more highly constrained surface to take as much advantage of the low weight bonds as a flatter surface. In addition, the random bonds couple to ∇h as well as to h modulo $b = 3$. An appropriate effective Hamiltonian is thus

$$H = \int d\mathbf{r} \left[\frac{K}{2} (\nabla h)^2 - \mathbf{f} \cdot \nabla h - w \cos\left(\frac{2\pi}{3} h - \gamma \right) \right] \quad (1)$$

with $\mathbf{f} \equiv \mathbf{f}(\mathbf{r})$ locally random with variance Δ, and $\gamma \equiv \gamma(\mathbf{r}) = 0, \pm\frac{2\pi}{3}$ with equal probability.

A related and well studied model (CDW) [10] has, instead, the $\{\gamma(\mathbf{r})\}$ being uniformly distributed random variables in $[0, 2\pi]$. While it is not clear a priori that these models are in the same universality class, both the following, and an RG analysis, imply they are: shifting $h(\mathbf{r})$ by $\frac{3}{2\pi}\gamma(\mathbf{r})$ changes the $\{f(\mathbf{r})\}$ introducing short range correlations that are different in the two models, but these should, be irrelevant on large scales of h and L. The CDW model has an elastic glass phase for $T < T_g$ below which w is relevant and renormalizes to a T-dependent fixed point while Δ grows as $\ln L$ yielding height variations $\overline{< [h(\mathbf{r}) - h(0)]^2 >} \approx \frac{b^2}{\pi} \Upsilon \ln^2 r$ [10].

An approximate functional RG analysis yields a similar structure at all temperatures with a universal $T = 0$ limit Υ_0 of the coefficient $\Upsilon(T)$ [11]. On large scales, the behavior is dominated by the competition between the random stressing $\mathbf{f}(\mathbf{r})$, and the stiffness K, with the random force correlations effectively $\overline{f_i(\mathbf{q}) f_j(-\mathbf{q})} = -C\delta_{ij} \ln q^2$ for small wave vectors q. We can thus work with the simple purely random-force limit of Eq. (1) with $w = 0$. We then immediately conclude that $\Upsilon_0 = \frac{C}{K^2 b^2}$.

In the presence of dislocations $h(\mathbf{r})$ becomes multivalued. It can be decomposed into two parts $h = h_e + h_d$, with h_e, the smooth elastic distortion while the singular function h_d, has a cut connecting the two dislocations at $\mathbf{r}_1$ and $\mathbf{r}_2$ with Burgers charge $b_1 = b = 3$ and $b_2 = -b = -3$ with $\nabla \times \nabla h_d = \sum_{i=1,2} b_i \delta^2(\mathbf{r} - \mathbf{r}_i)$ and $\nabla^2 h_d = 0$. The energy of a dislocation pair is

$$E_d = \frac{K}{2} \int d^2\mathbf{r} |\nabla h_d|^2 - \int d^2\mathbf{r} (\mathbf{f}^T \cdot \nabla h_d) \quad (2)$$

$$\approx \frac{Kb^2}{2\pi} \ln |\mathbf{r}_1 - \mathbf{r}_2| - b\left[g(\mathbf{r}_1) - g(\mathbf{r}_2) \right] \quad . \quad (3)$$

where the static random force field has been decomposed into longitudinal $\mathbf{f}^L = \nabla u(\mathbf{r})$ and transverse $\mathbf{f}^T = \nabla \times g(\mathbf{r})$ components. Since $u(\mathbf{r})$, $g(\mathbf{r})$, and ∇h_d are continuous across the cut while h_d jumps by b, $\mathbf{f}^L$ makes no contribution to E_d and Eq.(3) follows by integration by parts. The first term is the standard elastic cost and the second the disorder gain so that $g(\mathbf{r})$ is the *potential* felt by the dislocations. Its variance is $S_g \equiv \overline{g(\mathbf{q}) g(-\mathbf{q})} = -Cq^{-2} \ln q^2$ so that the elastic surface without dislocations and the dislocation potential have the same statistics after a rescaling: $S_h \equiv \overline{h(\mathbf{q}) h(-\mathbf{q})} \approx S_g / K^2$.

In terms of the statistical properties of the dislocation potential our numerical results can be understood. We first discuss fixed dislocation pairs. It has been argued [11] from a statistical symmetry (of the CDW model) that on large scales $f(\mathbf{r})$ and hence the potential $g(\mathbf{r})$ and E_d are gaussian. The shape of our computed defect energy distribution, its mean and its variance all agree with predictions from Eq. (3), $\overline{E_d} \approx (Kb^2/2\pi) \ln L$ —an exact result for the CDW model—and $\sigma(E_d) \approx \sqrt{b^2 C/\pi} \ln L$ with $K = 126(2)$ and $C = 6174(93)$ yielding $C/K^2 = 0.389(11)$. We also measured the variance of the height of the surface without dislocations finding a good fit to $\sigma^2(h) = 0.061(2) \ln^2 L + 0.477(7) \ln L + 0.765(13)$. The predicted coefficient of the $\ln^2 L$ is $C/2\pi K^2$ yielding $C/K^2 = 0.387(12)$. The agreement between the two estimates of C/K^2 further supports the validity of the random force model, including, in particular, the equality between the longitudinal and transverse parts of $\mathbf{f}(\mathbf{r})$

We now turn to optimized dislocations. If the first term in Eq. (3) can be ignored, the energy of a dislocation pair will be lowest if the dislocations are at the minimum (g_{min}) and maximum (g_{max}) of the random potential, $g(\mathbf{r})$. The distribution of the extrema of potentials like $g(\mathbf{r})$ whose variance grows as a power of $\ln L$, can be semi-quantitatively understood by thinking of iteratively optimizing over each factor of two in length scale, with the component of the random potential on scale l being essentially the contribution from Fourier components with $\frac{\sqrt{2}\pi}{l} < |q_x|, |q_y| < \frac{2\sqrt{2}\pi}{l}$. If scale l gives rise to a contribution to the variance of g of order $(\ln l)^{2\alpha}$—with $\alpha = \frac{1}{2}$ in our case—, then a *typical* $g(\mathbf{r})$ is the incoherent logarithmic sum over all scales, i.e. of order $(\ln l)^{\alpha+1/2}$. The maximum of g can be found, heuristically, by maximizing over the four points at scale $l = 1$ in each square of scale 2, then maximizing over the four scale 2 maxima in each scale 4 square, etc. Since the scale l structure of g is weakly correlated over scales much longer than l, a crude approximation is to ignore these correlations whereby one stage of optimization adds of order $(\ln l)^{\alpha}$ to the local scale l maximum [12]. Thus scales should be summed over *coherently* yielding $g_{max} \sim (\ln L)^{\alpha+1}$. From this hierarchical construction, it can be seen that the variance of $g_{\max}$ is dominated by the largest scales, so that $\sigma(g_{\max}) \sim (\ln L)^{\alpha}$. In our case, we thus expect $\overline{g_{\max} - g_{\min}} \sim (\ln L)^{3/2}$ which indeed dominates over the $\ln L$ elastic energy term in Eq. (3). Hence we expect $\overline{E_d^{\min}} \approx -Ab\sqrt{C}(\ln L)^{3/2}$ and $\sigma(E_d^{\min}) \approx Bb\sqrt{C}(\ln L)^{1/2}$ with some coefficients A and B.

The linear fits in Fig.3(b) give $A \approx 1.23(1)$ and $B \approx 0.56(1)$ using the C computed earlier. If the elastic part $\frac{Kb^2}{2\pi} \ln L$ is subtracted from $\overline{E_d^{\min}}$ by fitting the difference $\overline{E_d^{\min}(L)} - \overline{E_d^{\min}(L/2)}$ to $-(3\ln 2/2)Ab\sqrt{C}(\ln L)^{1/2} + const.$, this yields a very comparable $A \approx 1.17(5)$. Extremal heights $\overline{h_e^{\min}} \equiv \overline{h_{min} - h_{max}}$ of the elastic surface without dislocations and optimal dislocation energies can also be used to extract K via $\overline{E_d^{\min}} - bK\overline{h_e^{\min}} \approx \frac{Kb^2}{2\pi} \ln L + const.$. This yields $K \approx 114(1)$ in not unreasonable agreement with the $K \approx 126(2)$ from the *fixed* dislocation pairs.

An upper bound on $\overline{g_{\max} - g_{\min}}$ can be simply ob-

tained by noting that $\mathrm{Prob}[g_{\max} > M] \leq \sum_{\mathbf{r}} \mathrm{Prob}[g(\mathbf{r}) > M]$ so that, with L^2 points, the median $g_{\max}$ is less than the M for which the right hand side, $L^2\mathrm{Prob}[g(\mathbf{r}) > M]$ for fixed $\mathbf{r}$, equals $\frac{1}{2}$. If $g(\mathbf{r})$ is gaussian sufficiently far into the tail of its distribution, this yields $\overline{g_{\max} - g_{\min}} \leq \sqrt{C}\sqrt{\frac{8}{\pi}}(\ln L)^{\frac{3}{2}}$ so that $A \leq \sqrt{\frac{8}{\pi}} = 1.5957....$ The hierarchical optimization described above suggests that A might saturate this bound. To test the universality of A and this issue, we have measured the distributions of the extrema of the heights of the FPL surface without dislocations and several simulated exactly gaussian random surfaces with different α. Good fits are found to $(\ln L/a_\alpha)^{1+\alpha}$ with a_α a cutoff, yielding $A_{\text{FPL heights}} \approx 1.45(3)$, and $A_{\text{gaussian}} \approx 1.517(3), 1.307(6), 1.168(6)$, and $1.064(5)$ for $\alpha = 0, 1/2, 1$, and $3/2$ respectively. Similarly from the variances we obtain $B_{\text{FPL heights}} \approx 0.67(2)$, and $B_{\text{gaussian}} \approx 0.475(4), 0.637(7), 0.730(9)$, and $0.814(4)$ for $\alpha = 0, 1/2, 1$, and $3/2$ respectively.

We first note that all the extracted A's satisfy $A \leq \sqrt{8/\pi}$, and there appears to be a systematic trend for $A(\alpha)$ to decrease as α increases; if this is true then it is most likely that A is strictly less than the bound for all $\alpha > -\frac{1}{2}$ (for $\alpha < -\frac{1}{2}$ gaussian surfaces the variance saturates for large L and the extrema grow as $\sqrt{\ln L}$). The values of A for the nominally $\alpha = \frac{1}{2}$ cases, 1.17-1.23, 1.45 and 1.307 for the optimal dislocations, extrema heights, and the gaussian surface respectively, differ by substantially more than the apparent statistical errors as do the B's, 0.56, 0.67, 0.637 respectively. But given the narrow range available of $\ln L$ in spite of a large range of sizes and, as importantly, the lack of understanding of corrections to scaling, these results are certainly consistent with universal values of A and B for $\alpha = \frac{1}{2}$. At this point, however, understanding whether this is in fact the case, and also whether A for gaussian surfaces depends on α, must wait for better theoretical understanding.

Overall, we have found rather good agreement for a variety of large scale quantities with the RG prediction of equivalence at long scales of the FPL model and a random force model. Although extracting reliable exponents of $\ln L$ is not possible (especially with logarithmic corrections to scaling) the fact that the *coefficients* and ratios between these—extracted several ways—are in reasonable agreement is a more stringent test. But even if the random force equivalence is *not* valid, the data of Fig.3 clearly indicate the instability of large systems to dislocation pairs. With no restrictions on their number, dislocations will proliferate thereby driving the elastic constant K to zero.

We conclude with an alternate way to understand the structure of excitations in the elastic glass, via a picture developed for the three-dimensional case [5]. The basic excitations from a ground state are fractal domain walls surrounding regions in which h changes by b. Their fractal dimension, d_w, will be the same as that for the forced open wall that connects a pair of dislocations (Fig.1(c)) for which we find $d_w = 1.28(3)$ for fixed pairs and 1.30(3) for optimized pairs. [These contrast strongly with the connecting *strings* in the loop model which have $d_s = 1.75(3)$ and 1.74(3) respectively, very close to the value in the non-random loop model [7]]. The energy of a scale L wall constrained only on scale L is predicted to vary by of order $\sqrt{\ln L}$ but have mean independent of L. The incoherent logarithmic addition over all scales then yields variations of the fixed-end open domain wall energy, of order $\ln L$ and a mean of the same order—as found. But if the end positions can adjust to lower the wall energy near the dislocation at each scale, the energies add up *coherently* resulting in the $-\ln^{3/2} L$ mean optimal dislocation pair energy with order $\sqrt{\ln L}$ around the mean variations being dominated by the largest scale, in an analogous way to the extrema of the random potential $g(\mathbf{r})$ of the random force picture. Since the defect energy in the domain wall picture is concentrated on the wall, while it is spread out over a region of area $O(L^2)$ in the random force model, it is surprising that these yield the same predictions! But the fact that our results agree well with the domain wall picture in $2D$ lends strong support to the validity of the analogous picture in the $3D$ case for which it has been used to conclude that the $3D$ elastic glass phase is stable to dislocation loops [5].

We thank J. Kondev and C.L. Henley for useful discussions. This work has been supported in part by the National Science Foundation via grants DMR 9630064, DMS 9304586 and Harvard University MRSEC.

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